Closed-Form Design of Maxflat $R$-Regular IIR $M$th-Band Filters

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1 Introduction

$M$th-band filters are an important class of digital filters and have found numerous applications in multirate signal processing systems, filter banks and wavelets [1]. This paper considers the design problem of maxflat $R$-regular IIR $M$th-band filters, and gives the closed-form expression for its filter coefficients. The filter coefficients are directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition of the maxflat $R$-regular $M$th-band filters via the blockwise waveform moments.

2 IIR $M$th-Band Filters

Let $h_n (0 \leq n < \infty)$ be an impulse response of IIR digital filter $H(z)$. If $H(z)$ is a $M$th-band filter, its impulse response is required to be exactly zero-crossing except for one point $K$, i.e.,

$$h_{K+nM} = \begin{cases} \frac{1}{M} & (m = 0) \\ 0 & (m = \pm 1, \pm 2, \ldots) \end{cases}, \quad (1)$$

where $K$ and $M$ are integers, and $K$ corresponds to the desired group delay in the passband.

$M$th-band filter is required to be lowpass, and the desired frequency response is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-jK\omega} & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \quad (2)$$

Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$, i.e., $\hat{h}_n = h_{n+K}$. The desired frequency response of $\hat{H}(z)$ becomes

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1 & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \quad (3)$$

By using the polyphase representation, we have

$$H(z) = \sum_{i=0}^{M-1} z^{-i} H_i(z^M), \quad (4)$$

where

$$H_i(z) = \frac{N_1^i}{n_0} \sum_{n=0}^{N_2^i} a_n z^{-n} / \sum_{n=0}^{N_2^i} b_n z^{-n}, \quad (5)$$

where $N_1^i, N_2^i$ are the degree of the numerator and denominator, respectively, $a_n, b_n$ are real filter coefficients, and $b_0^i = 1$. Assume that $K = L_1 + L_2$, where $L_1, L_2$ are integers, and $0 \leq L_2 \leq M - 1$, it can be seen from the time-domain condition in (1) that $H_{L_2}(z) = z^{-L_1}/M$. Therefore, we have

$$\hat{H}(z) = z^K H(z) = \frac{1}{M} + \sum_{i=0}^{M-1} z^{-K-i} H_i(z^M). \quad (6)$$

It can be seen from (6) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{k=0}^{M-1} \hat{H}(e^{j(\omega + \frac{2k\pi}{M})}) = 1, \quad (7)$$

which means that the sum of the responses at the frequency points $\omega_k = \omega + 2k\pi/M$ for $k = 0, 1, \ldots, M - 1$ keep constant, regardless of what the filter coefficients are. From (7), we get

$$\hat{H}(e^{j\omega_0}) = 1 - \sum_{k=1}^{M-1} \hat{H}(e^{j\omega_k}). \quad (8)$$

It is clear that the frequency response at $\omega_0$ is dependent on the frequency responses at $\omega_k (k = 1, 2, \ldots, M - 1)$. If its stopband response is 0, then the frequency response of $\hat{H}(z)$ will become 1 in the passband, i.e., the magnitude response of $\hat{H}(z)$ is 1, and the group delay is $K$ in the passband. Therefore, the design problem of IIR $M$th-band filters with an arbitrarily specified $K$ can be reduced to the minimization of the stopband error of $\hat{H}(z)$.

3 Maxflat $R$-Regular IIR $M$th-Band Filters

In [5], the blockwise waveform moment around $K$ for $h_n$ is defined by

$$m_r(i) = \sum_{m=0}^{\infty} (mM + i - K)^r h_{m, M+i}, \quad (9)$$

where $0 \leq i \leq M - 1$. It follows from the definition of $m_r(i)$ that

$$\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r} \bigg|_{\omega = \frac{2km\pi}{M}} = (-j)^r \sum_{i=0}^{M-1} m_r(i) e^{-j(2i-2iK)\pi}, \quad (10)$$

i.e., the blockwise waveform moments describe the derivative behaviors of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M (0 \leq k \leq M - 1)$. It is seen in (10) that the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at $\omega_k = 2k\pi/M$ are the $M$-point DFT (Discrete Fourier Transform) of the blockwise waveform moments $m_r(i)$. Thus, by the inverse transform, we have

$$m_r(i) \leftarrow \frac{2r}{M} \sum_{k=0}^{M-1} \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r} \bigg|_{\omega = \frac{2km\pi}{M}} e^{\frac{2i(2i-K)\pi}{M}}. \quad (11)$$

It is clear that the blockwise waveform moments $m_r(i)$ bridge between the time and frequency domains by (9) and (11). Given the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$, the rth blockwise waveform moments $m_r(i)$ can be calculated via the IDFT in (11).
It is known in [1] that an \( M \)-th band filter is said to be \( R \)-regular if it has
\[
H(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})^R Q(z),
\]
where \( Q(z) \) is an IIR filter. It is equivalent to
\[
\frac{\partial H(e^{j\omega})}{\partial \omega} \bigg|_{\omega = \frac{\pi}{N}} = 0,
\]
for \( k = 1, 2, \cdots, M-1 \) and \( r = 0, 1, \cdots, R-1 \). It is obtained from (8) and (13) as
\[
M \sum_{n=0}^{N_i} (n M + i - K) a_n = \sum_{n=1}^{N_2} (n M)^r b_n = 0.
\]
From the relationship between \( H(z) \) and \( H(z) \) in (6), it is clear that \( H(z) \) has flat magnitude and group delay responses at \( \omega = 0 \) simultaneously.

By using (11), the blockwise waveform moments are obtained from the regularity condition in (13) and (14) as
\[
m_r(i) = \begin{cases} 
1 & (r = 0) \\
0 & (r = 1, 2, \cdots, R - 1)
\end{cases}.
\]

It can be obtained from (5) that
\[
K_i^{-1} H_i(z^M) = \sum_{n=0}^{N_2} b_n z^{-nM} = \sum_{m=0}^{\infty} b_{m+2} z^{-mM}.
\]

Next, we define the waveform moments for the numerator and denominator in (18) by
\[
\begin{align*}
m_r^N(i) &= \sum_{n=0}^{N_1} (n M + i - K) a_n \\
m_r^D(i) &= \sum_{n=0}^{N_2} (n M)^r b_n
\end{align*}
\]
Therefore, it can be seen by taking \( r \)th derivatives of (18) and substituting \( z = 1 \) that the condition in (17) becomes
\[
M m_r^N(i) = m_r^D(i) \quad (r = 0, 1, \cdots, R - 1).
\]

From the definition of \( m_r^N(i), m_r^D(i) \) in (19) and \( b_0 = 1 \), we obtain
\[
M \sum_{n=0}^{N_1} (n M + i - K)^r a_n - \sum_{n=1}^{N_2} (n M)^r b_n = \delta(r),
\]
where \( r = 0, 1, \cdots, R - 1 \), and
\[
\delta(r) = \begin{cases} 
1 & (r = 0) \\
0 & (r \neq 0)
\end{cases}.
\]

It should be noted that the coefficient matrix in (21) is the Vandermonde matrix with distinct elements. There is always a unique solution if \( R = N_1 + N_2 + 1 \). By using the Cramer’s rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde’s determinants. Therefore, a closed-form solution is obtained as
\[
\begin{align*}
a_n &= \frac{(-1)^{n+1} N_1! \prod_{m=0}^{N_1} (m + i - K)}{\prod_{m=0}^{N_1} (m - n + K - i)} \\
b_n &= \frac{(-1)^n N_2! \prod_{m=0}^{N_2} m + i - K}{\prod_{m=0}^{N_2} m - n + i - K}
\end{align*}
\]

Once \( M, K, N_1, N_2 \) are given, a set of filter coefficients \( a_n \) and \( b_n \) can be easily calculated by using (23). It is seen that besides \( N_1 + N_2 = R - 1 \) must be satisfied, it is possible for \( H(z) \) to have different \( N_1 \) and \( N_2 \) for \( 0 \leq i \leq M - 1 \).

**4 Conclusion**

In this paper, we have proposed a new closed-form solution for the maxflat \( R \)-regular IIR \( M \)-th band filters. The filter coefficients have been directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition via the blockwise waveform moments.

**References**


