Hilbert Transform Pairs of Orthonormal Symmetric Wavelet Bases 
Using Allpass Filters

Xi ZHANG

Department of Information and Communication Engineering
The University of Electro-Communications

1 Introduction

Hilbert transform pairs of wavelet bases have been proposed and proven to be successful in many signal and image processing applications [1], [2]. In this paper, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases using allpass filters.

2 Hilbert Transform Pairs of Wavelet Bases

It is well-known that orthonormal wavelet bases can be generated by two-band orthonormal filter banks \( \{H_i(z), G_i(z)\} \) \((i = 1, 2)\), where \(H_i(z)\) are assumed to be lowpass filter, and \(G_i(z)\) are highpass. The dilation and wavelet equations give the scaling and wavelet functions;

\[
\begin{align*}
\phi_i(t) &= \sqrt{2} \sum_n h_i(n) \phi_i(2t - n), \\
\psi_i(t) &= \sqrt{2} \sum_n g_i(n) \phi_i(2t - n),
\end{align*}
\]

(1)

where \(h_i(n)\) and \(g_i(n)\) are the impulse responses of \(H_i(z)\) and \(G_i(z)\), respectively. It is known in [2] that two wavelet functions \(\psi_1(t)\) and \(\psi_2(t)\) form a Hilbert transform pair;

\[
\psi_2(t) = H(\psi_1(t)),
\]

(2)

that is

\[
\Psi_2(\omega) = \begin{cases} -j \Psi_1(\omega) & (\omega > 0) \\ j \Psi_1(\omega) & (\omega < 0) \end{cases},
\]

(3)

if and only if two lowpass filters satisfy

\[
H_2(e^{j\omega}) = \frac{1}{\sqrt{2}} [A_c(z) + \tilde{A}_c(z)]
\]

(4)

where \(\Psi_1(\omega)\) are the Fourier transform of \(\psi_1(t)\).

3 Orthonormal Symmetric Solution

In this section, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases. Firstly, we use a complex allpass filter \(A_c(z)\) of order \(2N_1\) to construct \(H_1(z)\) and \(G_1(z)\) [3];

\[
\begin{align*}
H_1(z) &= \frac{1}{\sqrt{2}} [A_c(z) + \tilde{A}_c(z)] \\
G_1(z) &= \frac{j}{\sqrt{2}} [A_c(z) - \tilde{A}_c(z)]
\end{align*}
\]

(5)

where

\[
A_c(z) = z^{-2N_1} e^{j\eta} \sum_{n=0}^{N_1-1} \sum_{n=0}^{N_1-1} a_n^2 e^{-2n} + j \sum_{n=0}^{N_1-1} \sum_{n=0}^{N_1-1} a_{n+1}^2 e^{-2n+1}
\]

(6)

where \(a_n^r = a_{2N_1-n}^r\) are real, \(\eta = \pm \pi/4\) for even \(N_1\) and \(\eta = \pm 3\pi/4\) for odd \(N_1\). \(A_c(z)\) has a set of coefficients that are complex conjugate with ones of \(A_c(z)\). It is seen that \(H_1(z)\) and \(G_1(z)\) have linear phase responses and satisfy the orthonormality condition. The maximally flat solution for \(H_1(z)\) and \(G_1(z)\) has been proposed in [3], and \(H_1(z)\) has \(2N_1\) zeros at \(z = -1\). Next, we use a real allpass filter \(A_r(z)\) of order \(N_2\) to construct \(H_2(z)\) and \(G_2(z)\) [4];

\[
\begin{align*}
H_2(z) &= \frac{1}{\sqrt{2}} [z^K A_r(z^2) + z^{-K-1} A_r(z^{-2})] \\
G_2(z) &= \frac{1}{\sqrt{2}} [z^K A_r(z^2) - z^{-K-1} A_r(z^{-2})]
\end{align*}
\]

(7)

where \(K\) is integer and \(A_r(z)\) is defined by

\[
A_r(z) = z^{-N_2} \sum_{n=0}^{N_2} a_n^r z^n - \sum_{n=0}^{N_2} a_n^r z^{-n},
\]

(8)

where \(a_n^r\) are real. It is known in [4] that \(H_2(z)\) and \(G_2(z)\) are orthonormal and have linear phase responses also. \(K\) must be chosen as \(K = 2(N_2 - 2k)\) or \(K = 2(N_2 - 2k) - 1\) for \(k = 0, 1, \ldots, N_2\). The maximally flat solution for \(H_2(z), G_2(z)\) has been given in [4], and \(H_2(z)\) has \(2N_2 + 1\) zeros at \(z = -1\). Therefore, it is clear that \(H_1(z)\) and \(H_2(z)\) have already satisfied the magnitude condition in Eq.(4). To satisfy the magnitude condition in Eq.(4) approximately, we choose \(N_1 = N_2\) or \(N_1 = N_2 + 1\) to ensure \(H_1(z)\) and \(H_2(z)\) to have a close number of zeros at \(z = -1\) as possible. Therefore, the resulting pairs of orthonormal symmetric wavelet bases have almost same degrees of regularity.

4 Conclusion

In this paper, we have proposed a pair of orthonormal symmetric wavelet bases that form the Hilbert transform. The Hilbert transform pairs proposed in this paper have been constructed by using complex and real allpass filters.

References


